Homework 1

Remember you are allowed to discuss with classmates (or an AI tool), but that you need to tell me who/what you discussed with + the final submitted writeup should be your own work.

If you solve a problem using a computer algebra system (e.g., Macaulay2 or Singular) that is allowed, just provide some code instead of just saying "by Macaulay2"! Also please do so in the "spirit" of the problem, i.e., don't just use the function `isFPure` :)

(1) Prove the following statement:

Proposition 1. *Let $(R, \mathfrak{m}, k)$ be a local ring containing a* coefficient field*, i.e., there exists an injection $\gamma : k \to R$ such that $\pi \circ \gamma$ is an isomorphism, where π is the natural surjection $\pi : R \to R/\mathfrak{m} \cong k$. Then number of minimal generators for $F^e_* R$ as an R-module is*

$$[k : k^{p^e}] \cdot \dim_k(R/\mathfrak{m}^{[p^e]})$$

(2) (Exercise 5 in Ma–Polstra) Prove that if R is essentially of finite type[1] over an F-finite field, then R is also F-finite. Do this via first proving each of the following three facts:

(a) If R is F-finite, then R/I is F-finite for all ideals I.

(b) If R is F-finite, then $W^{-1}R$ is F-finite for all multiplicative sets W.

(c) If R is F-finite, then $R[x]$ and $R[[x]]$ are F-finite for an indeterminate x.

(3) (Exercise 1 in Ma–Polstra) Prove that if there exists an $e_0 > 0$ such that $F^{e_0}_* R$ is a finite R-module, then in fact $F^e_* R$ is a finite R-module for all $e > 0$.

(4) Let $R = \mathbb{F}_2[x^2, x^3]$. Prove that R is *not* F-split.[2][3]

[1] essentially of finite type = a localization of something of finite type

[2] Hint: You are welcome to use the generators & relations for $F_* R$ we found in class, no need to reprove!

[3] Note: This problem is definitely doable by hand using what we've learned as of 1/22, but you are also welcome to use any other methods for testing F-splitting that we learn in-class between now and when this HW is due.