

## Homework 2

Remember you are allowed to discuss with classmates (or an AI tool), but that you need to tell me who/what you discussed with + the final submitted writeup should be your own work.

(1) Let  $R$  be a ring and let  $f_1, \dots, f_t$  be a regular sequence.

(a) Prove the following lemma:

**Lemma 1** (Colon Capturing). *For all  $0 \leq i < t$ , we have*

$$\langle f_1, \dots, f_i \rangle : \langle f_{i+1} \rangle = \langle f_1, \dots, f_i \rangle,$$

where when  $i = 0$  the left side of the colon is the ideal generated by NO elements, i.e., the zero ideal.

(b) Using the above lemma and other facts about regular sequences<sup>1</sup> and colon ideals<sup>2</sup>, prove the following proposition:<sup>3</sup>

**Proposition 1.** *Let  $f_1, \dots, f_t$  be a permutable regular sequence (for example, either any regular sequence in a local ring, or a homogeneous regular sequence in a graded ring). Assume that  $R$  has char  $p > 0$ . Let  $I = \langle f_1, \dots, f_t \rangle$ . Then*

$$I^{[p^e]} : I = I^{[p^e]} + \langle (f_1 \cdots f_t)^{p^e - 1} \rangle.$$

(2) (a) Let  $S$  be an  $F$ -finite regular local ring, let  $\{F_* b_i\}_{i=1}^t$  be basis for  $F_* S$ , and let  $\Phi \in \text{Hom}_S(F_* S, S)$  be the generating map. Suppose that  $g \in S$  can be written as  $g = \sum_i a_i^p b_i$  for some  $a_i \in S$ . Show that

$$\Phi(F_*(\langle g \rangle)) = \langle a_1, \dots, a_t \rangle.$$

(b) The *Frobenius root* of an ideal  $I$  (written  $I^{[1/p]}$ ) is the smallest ideal  $J$  such that  $I \subset J^{[p]}$ . Prove that in the setup of part (a), we have

$$\langle g \rangle^{[1/p]} = \langle a_1, \dots, a_t \rangle.$$

(3) Let  $S = k[x_1, \dots, x_n]$  for an  $F$ -finite field  $k$  and let  $f$  be a non-zero homogeneous polynomial of degree  $d$ . Using Fedder's criterion<sup>4</sup> prove that if  $d > n$ , then  $S/f$  is not  $F$ -split.

<sup>1</sup>You may use without proof the fact that for any regular sequence, powers of regular sequences are still regular. (In the local case this is Lemma 1.52 of the 906 notes, but here is a reference to the more general fact.)

<sup>2</sup>You may use any of the properties on Wikipedia without proof!

<sup>3</sup>Hint: You may find it helpful to first prove that for a permutable regular sequence and powers  $0 \leq b_i \leq a_i$  for all  $1 \leq i \leq t$ , that  $\langle f_1^{a_1}, \dots, f_t^{a_t} \rangle : \langle \prod_{i=1}^t f_i^{b_i} \rangle = \langle f_1^{a_1 - b_1}, \dots, f_t^{a_t - b_t} \rangle$ .

<sup>4</sup>Recall that Fedder also applies for homogeneous ideals in a (standard) graded ring, taking  $\mathfrak{m}$  = the homogeneous maximal ideal

(4) Let  $k$  be an  $F$ -finite field and let  $R = k[[x, y, z]]/\langle x^2 + y^3 + z^7 \rangle$ . Prove that  $R$  is never  $F$ -split in any characteristic.

(5) Let  $X$  be a  $4 \times 4$  matrix of indeterminates, and let  $S = \mathbb{Z}/3[X]$  be the polynomial ring over these indeterminates (so, a 16 variable polynomial ring). Let  $I_2$  be the ideal of  $2 \times 2$  minors. Using Macaulay2 and Fedder's criterion, verify that  $S/I_2$  is  $F$ -split.<sup>5</sup>

*Allowed Functions:* The only thing from `TestIdeals` you are allowed to use is the `frobenius` function (so, no `isFPure!`). However, you are free to use ANY other functions available in base Macaulay2 or any other packages.<sup>6</sup>

*Submission:* Turn in your M2 code, HCC `.submit` file (if you used one), AND the output to your code. Format flexible: can be a screenshot if you did it in interactive mode, or copy-pasting the file contents in a verbatim environment into `tex`, or whatever works best for you.

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<sup>5</sup>For me on the HCC, this check took about 1 hour, and used about 5gb of memory. If you want to double-check your code for errors, first try the much-faster  $p = 2$  version (which is also  $F$ -split!)

<sup>6</sup>Explore the documentation... you might find something useful, like `minors`, or how to quickly make a ring with 16 variables!