Homework 3

Remember you are allowed to discuss with classmates (or an AI tool), but that you need to tell me who/what you discussed with + the final submitted writeup should be your own work.

(1) Suppose that $S = R_1 \times R_2$, and that R_1, R_2, and S are all F-finite[1]. Prove that S is strongly F-regular if and only if both R_1 and R_2 are strongly F-regular[2].

(2) (Exercise 13 of [MP]) Let $R \to S$ be a faithfully flat extension of F-finite rings.

(a) Let M, N be finitely generated R-modules, and let $\psi : M \to N$ be an R-module map. Prove that ψ is surjective if and only if $\psi \otimes_R \mathrm{id}_S : M \otimes_R S \to N \otimes_R S$ is surjective.

(b) Prove that if c is a non-zerodivisor on R, then c is also a non-zerodivisor on S.

(c) Prove that if S is strongly F-regular, then so is R.

(3) (Glassbrenner's Criterion). Let $(S, \mathfrak{m}, k)$ be an F -finite regular local ring of prime characteristic $p > 0$ and let $I \subseteq S$ be an ideal[3]. Let $R = S/I$.

(a) Let $c \in S$. Prove that R is e-F-split along c if and only if $c(I^{[p^e]} : I) \not\subseteq \mathfrak{m}^{[p^e]}$.[4]

(b) Conclude that the following are equivalent:

(i) R is strongly F-regular.

(ii) For every $c \in S$ not in any minimal prime of I, there exists $e > 0$ such that $c \notin \mathfrak{m}^{[p^e]} : (I^{[p^e]} : I)$.

(iii) $\bigcap_{e \geq 0} \mathfrak{m}^{[p^e]} : (I^{[p^e]} : I)$ is contained in the union of the minimal primes.[5]

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[1] In fact this result holds without the F-finiteness hypothesis (and both R_i F-finite automatically implies S F-finite), but I am including it since for us F-finiteness is part of the definition of strong F-regularity

[2] Hint: There are multiple ways to approach this. One approach is mentioned in the proof of Theorem 3.9 of [MP], which you can look at if you get stuck!

[3] Note: Just like Fedder, Glassbrenner's criterion also holds if S is a standard graded ring over an F-finite field, $\mathfrak{m}$ is the homogeneous maximal ideal, and I is a homogenous ideal

[4] Hint: Use results from Jan 29th & Feb 3rd! And try to modify the proof of Fedder's criterion.

[5] Fun fact: By prime avoidance, you could even restate this a fourth way and require that the intersection by contained in a single minimal prime!

(4) Prove the following number theory results. Since (a) is a special case of (b), you only need to write up a solution to (b). However, you may find it helpful to think about the simpler part (a) first!

(a) (Lucas's Theorem) Write $n = \sum_{i=0}^{d} n_i p^i$ and $m = \sum_{i=0}^{d} m_i p^i$ where $0 \le n_i, m_i < p$ for all i.[6] Then

$$\binom{n}{m} \equiv \prod_{i=0}^{d} \binom{n_i}{m_i} \mod p.$$

Here we follow the convention that $\binom{a}{b} = 0$ if $b > a$.

(b) (Multinomial Lucas's Theorem) Suppose that $n = m_1 + \cdots + m_t$. Write $n = \sum_{i=0}^{d} n_i p^i$ and $m_j = \sum_{i=0}^{d} m_{i,j} p^i$ where $0 \le n_i, m_{i,j} < p$ for all i. Then

$$\binom{n}{m_1, \ldots, m_t} \equiv \prod_{i=0}^{d} \binom{n_i}{m_{i,1}, \ldots, m_{i,t}} \mod p.$$

Here we follow the convention that $\binom{a}{b_1, \ldots, b_t} = 0$ if $b_1 + \cdots + b_t \ne a$.

(5) (Exercise 17 of [MP]) Let k be an algebraically closed[7] field and let

$$R = k[x_1, \ldots, x_d]/\langle x_1^n + \cdots + x_d^n \rangle.$$

Using Glassbrenner's criterion, prove each of the following facts:

(a) Show that R is not strongly F -regular if $n \ge d \ge 2$.

(b) Show that R is strongly F-regular if $n < d$ and $p \gg 0$.[8]

[6]Hint: Both parts (a) & (b) are still true if you allow the top coefficient to be even larger, i.e., if we relax the last inequality to just be $n_d, m_d, m_{d,i} \ge 0$. You may actually find it easier to prove this more general version!

[7]This problem is also true with the weaker hypothesis that k is F-finite. You would prove that from the algebraically closed case by using Problem 2 and the fact that if $R \to S$ is faithfully flat, so is $R/I \to S/IS$.

[8]Hint: Use "test element" criterion (Thm 3.11 of [MP]; done in-class on Tuesday 02/24); an algebraic geometry criterion for non-singularity; and use some number theory to make divisibility things nicer.