Homework 4

Remember you are allowed to discuss with classmates (or an AI tool), but that you need to tell me who/what you discussed with + the final submitted writeup should be your own work.

(1) (Exercise 4.14 of [SS]) Write $K(R)$ for the total ring of fractions. A reduced Noetherian ring R of characteristic $p > 0$ is called *weakly normal* if $x \in K(R)$ and $x^p \in R$ implies that $x \in R$. Any ring is called *seminormal* if $x \in K(R)$, and $x^2, x^3 \in R$ implies that $x \in R$.

  (a) Show that Frobenius split rings are weakly normal, WITHOUT using the fact that F-injective rings are weakly normal.[1]

  (b) Show that weakly normal rings are seminormal.

(2) (Exercise 4.12 of [SS]) Let R be a Noetherian char p ring. Suppose that $\varphi : F^e_* R \to R$ is an R-linear map.

  (a) Fix I an ideal. Prove that $\varphi(F^e_*(\Gamma_I(R))) \subset \Gamma_I(R)$, where $\Gamma_I(R) = \bigcup_{t\in\mathbb{N}} \operatorname{ann}(I^t)$.

  (b) Suppose further that R is reduced. Prove that if $Q \subset R$ is a minimal prime of R, then $\varphi(F^e_* Q) \subset Q$.

(3) (Exercise 23 of [MP]) Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring of char p and $d = \dim R$.

  (a) Prove that $H^d_{\mathfrak{m}}(R) \neq 0$. [2]

  (b) Prove that the kernel of the natural Frobenius action on $H^d_{\mathfrak{m}}(R)$ is a proper submodule of $H^d_{\mathfrak{m}}(R)$.

(4) (Exercise 7.15 of [SS]) Let $(R, \mathfrak{m}, k)$ be a local Cohen-Macaulay ring[3] of dimension d. Prove that TFAE:

  (a) The natural Frobenius action on $H^d_{\mathfrak{m}}(R)$ is injective (in other words, R is F-injective)

  (b) For every s.o.p. $f_1, \ldots, f_d$, the ideal $I = \langle f_1, \ldots, f_d\rangle$ has

$$I = \langle g \mid g^p \in I^{[p]}\rangle.$$

  (c) There exists an s.o.p. $f_1, \ldots, f_d$, such that the ideal $I = \langle f_1, \ldots, f_d\rangle$ has[4]

$$I = \langle g \mid g^p \in I^{[p]}\rangle.$$

[1] ...because that fact makes this problem boring.

[2] Hint: You may find it helpful to use the Theorem from Tuesday 03/10, as well as to use facts about completion from 906 (especially facts from Section 3.4 of this semester's notes!)

[3] Fact (you may use without proof; possibly helpful for (c) implies (a)): If R is a CM ring and $\underline{x}$ and $\underline{y}$ are sop's, then $\dim_k(\frac{\langle x_1,\ldots,x_d\rangle:\mathfrak{m}}{\langle x_1,\ldots,x_d\rangle}) = \dim_k(\frac{\langle y_1,\ldots,y_d\rangle:\mathfrak{m}}{\langle y_1,\ldots,y_d\rangle})$. This number is the *Cohen-Macaulay type* of R.

[4] Hint: Prove for any zero-dimensional local ring $(S, \mathfrak{m})$ that the extension $(0 : \mathfrak{m}) \hookrightarrow S$ is essential. Then use this to prove that for a local ring $(R, \mathfrak{m})$ of ANY dimension d and an sop $f_1, \ldots, f_d$, that the map $\left\{ \left[\frac{r}{f_1^t\cdots f_d^t}\right] \,\middle|\, t \in \mathbb{N},\ r \in \langle f_1^t, \ldots, f_d^t\rangle : \mathfrak{m} \right\} \hookrightarrow H^d_{\mathfrak{m}}(R)$ is essential. (Or, just do the variant of this where you restrict to the submodules where your denominators use a fixed power of t in both source and target.)