Homework 5

Remember you are allowed to discuss with classmates (or an AI tool), but that you need to tell me who/what you discussed with + the final submitted writeup should be your own work.

(1) In this question we see some generally useful comm alg facts[1]. In particular, we are NOT assuming char p! First, some definitions:

Definition 1. Let A be a Noetherian ring and let $k \in \mathbb{N}$.

- A satisfies *Serre's condition* (R_k) if for all primes $\mathfrak{p}$ of height $\leq k$, we have $A_\mathfrak{p}$ is a regular local ring.
- A satisfies *Serre's condition* (S_k) if for all primes $\mathfrak{p}$, we have $\operatorname{depth}(A_\mathfrak{p}) \geq \min\{k, \operatorname{ht}(\mathfrak{p})\}$.

Now for the problem:

(a) Give an example of a ring that has (S_1) but is not reduced.

(b) Give an example of a ring that has (R_0) but is not reduced.

(c) Prove that A is reduced if and only if A satisfies (R_0) and (S_1).

(d) Prove that A is Cohen-Macaulay if and only if A satisfies (S_d) for $d = \dim A$, if and only if A satisfies (S_k) for all k.[2]

(2) In this question, we are back to assuming A has char p.

(a) Let $(A, \mathfrak{m})$ be a local char $p > 0$ ring of dimension 0. Prove that A is F-injective if and only if it is reduced.

(b) Let $(A, \mathfrak{m})$ be a local char $p > 0$ ring of dimension > 0. Prove that if A is F-injective then $H^0_\mathfrak{m}(A) = 0$.

(c) Let A be an F-injective ring. Prove that A is reduced.

(3) (Exercise 22 of [MP]) Let R be an $\mathbb{N}$-graded ring over a field of prime characteristic $p > 0$ with homogeneous maximal ideal $\mathfrak{m}$. Show that

(a) If R is F-injective, then $[H^i_\mathfrak{m}(R)]_{>0} = 0$ for each i. Conclude that $a(R) \leq 0$.

(b) If R is F-rational, then $[H^d_\mathfrak{m}(R)]_{\geq 0} = 0$. Conclude that $a(R) < 0$.

[1] Not included is the useful but harder to prove fact that A is normal if and only if A satisfies (R_1) and (S_2).

[2] This has nothing to do with anything else on this HW, but is such a cute fact that I'm including it anyways.

(4) Here are some more useful colon ideal facts:

(a) Let I unmixed ideal (i.e., no embedded primes/all associated primes of I are minimal over I) of a regular local ring $(R, \mathfrak{m})$. Prove that [3]

$$\mathrm{Ass}(I^{[p^e]} : I) \subseteq \mathrm{Ass}(I).$$

Conclude that if I is $\mathfrak{p}$-primary then $I^{[p^e]} : I$ is $\mathfrak{p}$-primary.

(b) Let $(R, \mathfrak{m})$ be a regular local ring and I unmixed. Let $I = \bigcap_{i=1}^n Q_i$. Prove that

$$I^{[p]} : I = \bigcap_i (Q_i^{[p]} : Q_i).$$

(5) As on Thurs 03/26, let $S = k[x, y, z, u, v]_{\mathfrak{m}}$, let

$$I = I_2 \begin{pmatrix} x^n & z & v \\ u & z & y^n \end{pmatrix} = \langle z(x^n - u), z(v - y^n), x^n y^n - uv \rangle$$

be the ideal of 2x2 minors, and let $R = S/I$.

(a) Compute the minimal primary decomposition for I.[4]

(b) Prove that if $p \leq n$ then R is NOT F-split.[5]

[3]Hint: First show that $\mathrm{Ass}(I) = \mathrm{Ass}(I^{[p^e]})$. For the hint and for $\mathrm{Ass}(I^{[p^e]} : I)$, think about the set of non-zero divisors

[4]Hint: Use the $\mathbb{Z}^3$-grading coming from the columns, so that $\deg x = e_1$, $\deg u = ne_1$, $\deg z = e_2$, and $\deg y = e_3$, $\deg v = ne_3$.
You may use without proof the fact that even in the multigraded setting, associated primes of homogeneous ideals are homogeneous.

[5]Hint: Modding out by a monomial ideal can make a complicated equation more understandable.