Worksheet on Fedder's Criterion

Part 1: Proving Fedder

Prove each of the following results:

Lemma 1. *If there exists a map* $\pi : F^e_* R \to R$ *such that* $\pi(F^e_* d) = 1$, *and* $c|d$ *in* R, *then there exists a map sending* $F^e_* c \mapsto 1$. *In particular, if there is ANY surjective map* $F^e_* R \to R$ *then* R *is* F*-split.*

Lemma 2. *Let* $\mathrm{ev}_d : \mathrm{Hom}_R(F^e_* R, R) \to R$ *be the "evaluation at* $F^e_* d$*" map, so that* $\mathrm{ev}_d(\varphi) = \varphi(F^e_* d)$. *Then there exists a map* $\pi \in \mathrm{Hom}_R(F^e_* R, R)$ *such that* $\pi(F^e_* d) = 1$ *if and only if* ev_d *is surjective.*

Proposition 1. *The following are equivalent for any* F*-finite ring* R*:*

(1) R *is* F*-split*

(2) $W^{-1}R$ *is* F*-split for all multiplicative sets* W.

(3) $R_{\mathfrak{p}}$ *is* F*-split for all prime ideals* $\mathfrak{p}$

(4) $R_{\mathfrak{m}}$ *is* F*-split for all maximal ideals* $\mathfrak{m}$

Corollary 1. *If* R *is* F*-finite then the locus of* F*-split points is open, and is specifically equal to* $\mathrm{Spec}\, R \setminus \mathbb{V}(\mathrm{im}\, \mathrm{ev}_1)$.

Lemma 3. *Let* S *be regular, take* $t \in I^{[p^e]} : I$ *and let* $\varphi = F^e_* t \cdot \Phi$. *Show that* $\Psi(\varphi)$ *(i.e.,* φ *viewed as a map on* S/I*) is surjective if and only if* $t \notin \mathfrak{m}^{[p^e]}$.

Part 2: Using Fedder

You may use the following two facts without proof (see HW2 for the first one!)

Proposition 2. *If $f_1, \dots, f_t$ is a regular sequence and $I = \langle f_1, \dots, f_t \rangle$ then*

$$I^{[p^e]} : I = I^{[p^e]} + \langle (f_1 \cdots f_t)^{p^e - 1}$$

Proposition 3. *If S is a graded ring with homogeneous maximal ideal $\mathfrak{m}$, then S is F-split if and only if $S_{\mathfrak{m}}$ is F-split; and Fedder's criterion holds for graded rings.*

(1) Let S be a polynomial ring and I a squarefree monomial ideal. Prove that S/I is F-split?

(2) Let $S = \mathbb{F}_p[x, y, z]$ and $I = \langle x^3 + y^3 + z^3$. For which values of p is S/I F-split?